AN ELECTROSTATIC INTERPRETATION OF ZEROS OF HERMITE-TYPE ORTHOGONAL POLYNOMIALS

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Abstract

Interpretation electrostatic of the distribution of the zeros of the Hermite-type orthogonal polynomials which are orthogonal with respect to a perturbation of the Hermite weight function by the addition of a mass point at zero. These polynomials satisfy a second order linear differential equation with polynomials coefficients. It plays an important role in the electrostatic interpretation of the distribution of the zeros of this family of polynomials.

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1 Introduction

The aim of this contribution is to apply the Stieltjes ideas, see [9] as well as [10] and [11], to study the electrostatic interpretation of the distribution of zeros of Hermite-type orthogonal polynomials. For absolutely continuous measures Ismail [6] proved that zeros of general orthogonal polynomials, under some integrability conditions for their weight functions, are the solution of an electrostatic equilibrium problem of n movable unit charges in the presence of an external potential. In particular, as an example he studied electrostatic of the zeros of the orthogonal polynomial sequence with respect to the Freud weight function $w(x) = e^{-x^4}$ supported on $\mathbb{R}$.

The case of measures with mass points outside or in the boundary of the support of the measure has been analyzed in [3], [4], and [7]. It is a natural question to ask about the electrostatic interpretation of the location of the zeros of the corresponding orthogonal polynomials when a perturbation of the linear functional $\mathcal{L}$, associated with the Hermite weight, is introduced. The perturbation is based on the addition of a Dirac linear functional supported at zero. This case has not been considered in the work by Ismail, [7]. Firstly, we obtain the relation between both orthogonal polynomial sequences as well as the explicit expression of the recursion coefficients of the new orthogonal polynomials. The main role is played by the second order linear differential equation that these polynomials satisfy since this yields the electrostatic interpretation. In section 3 two operators associated with our orthogonal polynomials will be obtained. In the next section we derive the corresponding differential equation through the previous two operators. Finally, in the last section the electrostatic interpretation will be developed.

2 Preliminaries

Let $\mathcal{U}$ be the linear functional

$$< \mathcal{U}, q(x) >= \int_{-\infty}^{\infty} q(x)e^{-x^2}dx + \lambda q(0), \qquad q(x) \in \mathbb{P}, \tag{1}$$

where $\lambda \in \mathbb{R}^+$, $\mathbb{P} = \mathbb{R}[x]$ is the linear space of polynomials with real coefficients, and $\mathbb{P}_n$ is the linear subspace of polynomials of degree at most n.

A polynomial sequence $\{\tilde{H}_n(x)\}_{n\geq 0}$, where $deg(\tilde{H}_n(x)) = n$ for $n \geq 0$, is said to be orthogonal with respect to the functional $\mathcal{U}$ if and only if:

$$< \mathcal{U}, \tilde{H}_n(x)\tilde{H}_m(x) >= \tilde{k}_n\delta_{n,m},$$

where $\tilde{k}_n > 0$, for $n, m \geq 0$, and as usual

$$\delta_{n,m} = \begin{cases} 0 & \text{for} \quad n \neq m, \\ 1 & \text{for} \quad n = m. \end{cases}$$

Such a family of polynomials is said to be a Hermite-type sequence.

Let $\mathcal{L}$ be the Hermite linear functional defined by

$$< \mathcal{L}, q(x) >= \int_{-\infty}^{\infty} q(x)e^{-x^2}dx, \qquad q(x) \in \mathbb{P}. \tag{2}$$

Consider $\{H_n(x)\}_{n\geq 0}$ the sequence of monic polynomials orthogonal with respect to $\mathcal{L}$. Then

$$< \mathcal{L}, H_n(x)H_m(x) >= k_n\delta_{n,m},$$

where $k_n > 0$ for $n, m \geq 0$. The Hermite linear functional is symmetric, (see [2]). Thus $\{H_n(x)\}_{n\geq 0}$ satisfies the three-term recurrence relation (TTRR)

$$xH_n(x) = H_{n+1}(x) + a_nH_{n-1}(x), \qquad n \geq 0,$$

where by a convention $H_{-1}(x) = 0$ and $a_n \in \mathbb{R}_+$ is

$$a_n = \frac{< \mathcal{L}, H_n^2(x) >}{< \mathcal{L}, H_{n-1}^2(x) >}, \qquad n \geq 1.$$

Moreover, (see [**?**])

$$a_n = \frac{n}{2}, \qquad n \geq 0. \tag{3}$$

Notice that both linear functionals are closely connected. Indeed

$$< \mathcal{U}, q(x) >=< \mathcal{L}, q(x) > +\lambda q(0), \qquad q(x) \in \mathbb{P}. \tag{4}$$

Let $\{\widehat{H}_n(x)\}_{n\geq 0}$ be the monic polynomial sequence orthogonal with respect to $\mathcal{U}$. Now we derive the three-term recurrence relation which the polynomial sequence $\{\widehat{H}_n(x)\}_{n\geq 0}$ satisfies. Since $\mathcal{U}$ is symmetric then the recurrence formula (see [2]) satisfied by $\{\widehat{H}_n(x)\}_{n\geq 0}$ is

$$x\widehat{H}_n(x) = \widehat{H}_{n+1}(x) + \widehat{a}_n \widehat{H}_{n-1}(x), \qquad n \geq 0. \tag{5}$$

From the three-term recurrence relation (5) we get

$$\begin{array}{lll} (i) & \widehat{H}_{2n}(0) = \widehat{H}'_{2n+1}(0) + \widehat{a}_{2n}\widehat{H}'_{2n-1}(0), & n \geq 0, \\ (ii) & \widehat{H}_{2n+2}(0) = -\widehat{a}_{2n+1}\widehat{H}_{2n}(0), & n \geq 0. \end{array}$$

From [8] we can deduce explicitly the recursion coefficients $\widehat{a}_n$. First we study the even coefficients

$$\begin{aligned} \widehat{a}_{2n} &= \frac{4^n\Gamma(n)^2 n + 4\Gamma(2n)n\lambda + 2\Gamma(2n)\lambda}{\Gamma(n)^2 4^n + 4\Gamma(2n)\lambda} \\ &= \frac{4^n(n-1)!^2 n + \left[4n(2n-1)! + 2(2n-1)!\right]\lambda}{4^n(n-1)!^2 + 4(2n-1)!\lambda} \\ &= \frac{2^{2n-1}n!(n-1)! + (2n+1)(2n-1)!\lambda}{2^{2n-1}(n-1)!^2 + 2(2n-1)!\lambda}. \end{aligned} \tag{6}$$

The odd coefficients are

$$\begin{aligned} \widehat{a}_{2n+1} &= \frac{(2n+1)n\left[64^n\Gamma(n)^2 + 16^n 4\Gamma(2n)\lambda\right]}{2\left[\Gamma(n)^2 n 64^n + 16^n 4n\Gamma(2n)\lambda + 16^n 2\Gamma(2n)\lambda\right]} \\ &= \frac{(2n+1)n\left[64^n(n-1)!^2 + 16^n 4(2n-1)!\lambda\right]}{2\left[(n-1)!^2 n 64^n + 16^n 4n(2n-1)!\lambda + 16^n 2(2n-1)!\lambda\right]} \\ &= \frac{(2n+1)n\left[2^{2n-1}(n-1)!^2 + 2(2n-1)!\lambda\right]}{2^{2n}n!(n-1)! + 2(2n+1)(2n-1)!\lambda}. \end{aligned} \tag{7}$$

Proposition 1 *The relation between the recursion coefficients is*

$$\widehat{a}_{2n+1}\widehat{a}_{2n} = \frac{(2n+1)n}{2}, \qquad n \geq 0.$$

Proof. It holds from (6) and (7). ∎

Notice that from the previous Proposition 1 and (3) we get

$$\widehat{a}_{2n+1}\widehat{a}_{2n} = a_{2n+1}a_{2n}, \qquad n \geq 0.$$

Furthermore from (6) and (7)

$$
\begin{aligned}
\widehat{a}_{2n} &= n\frac{2^{2n-1}(n-1)!^2+\dfrac{(2n+1)(2n-1)!}{n}\lambda}{2^{2n-1}(n-1)!^2+2(2n-1)!\lambda} \\
&= n\frac{1+\dfrac{2n+1}{2n}\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda} \\
&= n\frac{1+\left(1+\dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda} \\
&= n\frac{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda+\dfrac{1}{2n}\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}
\end{aligned}
$$

Then

$$
\begin{aligned}
\widehat{a}_{2n} &= n\left[1+\frac{\dfrac{1}{2n}\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}\right] \\
&= n\left[1+\frac{\dfrac{1}{2n}\left(1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda-1\right)}{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}\right] \\
&= n\left[1+\frac{1}{2n}-\frac{\dfrac{1}{2n}}{1+\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}\right].
\end{aligned}
$$

For the odd coefficients we get

$$
\begin{aligned}
\widehat{a}_{2n+1} &= \frac{(2n+1)n\left[2^{2n-1}(n-1)!^2+2(2n-1)!\lambda\right]}{2^{2n}n!(n-1)!+2(2n+1)(2n-1)!\lambda} \\
&= \frac{2n+1}{2}\frac{2^{2n-1}(n-1)!^2+2(2n-1)!\lambda}{2^{2n-1}(n-1)!^2+\frac{2n+1}{n}(2n-1)!\lambda} \\
&= \frac{2n+1}{2}\frac{1+\frac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1+\frac{2n+1}{2n}\frac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}.
\end{aligned}
$$

On the other hand, we introduce the kernel polynomials

$$
K_n(x,y)=\sum_{j=0}^{n}\frac{H_j(x)H_j(y)}{<\mathcal{L},H_j^2(x)>}.
$$

Proposition 2 *For $n \geq 0$,*

$$
\begin{aligned}
&(i) \qquad \widehat{H}_{2n+1}(x) = H_{2n+1}(x), \\
&(ii) \qquad \widehat{H}_{2n}(x) = H_{2n}(x) - \lambda \frac{H_{2n}(0)K_{2n-2}(x,0)}{1+\lambda K_{2n-2}(0,0)}.
\end{aligned}
$$

Proof. If we consider the Fourier expansion

$$
\widehat{H}_n(x) = H_n(x) + \sum_{k=0}^{n-1} a_{n,k} H_k(x), \qquad n \geq 0,
$$

then

$$
a_{n,k} = \frac{< \mathcal{L}, \widehat{H}_n(x) H_k(x) >}{< \mathcal{L}, H_k^2(x) >} = -\frac{\lambda \widehat{H}_n(0) H_k(0)}{< \mathcal{L}, H_k^2(x) >}, \qquad 1 \leq k \leq n-1.
$$

Thus

$$
\widehat{H}_n(x) = H_n(x) - \lambda \widehat{H}_n(0) K_{n-1}(x,0), \qquad n \geq 1.
$$

The evaluation at zero of the last expression yields

$$
\widehat{H}_n(0) = \frac{H_n(0)}{1+\lambda K_{n-1}(0,0)}. \tag{8}
$$

Therefore the statements of the Proposition follows in a straightforward way. ∎

3 Differential recurrence relation

We are interested in finding a second order linear differential equation satisfied by the orthogonal polynomials $\widehat{H}_n$. To obtain it we will use the fact that the linear functional $\mathcal{U}$ is semiclassical, that is, it satisfies the following distributional Pearson equation

$$
D(x^2 \mathcal{U}) = 2x(1-x^2)\mathcal{U},
$$

where D denotes the derivative operator.

3.1 Lowering operator

Theorem 3 *$\widehat{H}_n(x)$ satisfies the following differential recurrence relation*

$$
x^2 \widehat{H}_n'(x) = A(x,n)\widehat{H}_{n-1}(x) - B(x,n)\widehat{H}_n(x), \qquad n \geq 1,
$$

where

$$
\begin{aligned}
A(x,n) &= \widehat{a}_n \Big[2x^2 + C_{n+1} + C_n \Big], && (9)\\
B(x,n) &= C_n x, && (10)
\end{aligned}
$$

and

$$
C_n = 2\widehat{a}_n - n, \qquad n \geq 1. \tag{11}
$$

Proof. Consider the Fourier expansion

$$
x^2 \widehat{H}_n'(x) = n \widehat{H}_{n+1}(x) + \sum_{j=0}^{n} \lambda_{n,j} \widehat{H}_j(x), \tag{12}
$$

where

$$\lambda_{n,j} = \frac{\langle \mathcal{U}, x^2 \widehat{H}'_n(x)\widehat{H}_j(x)\rangle}{\langle \mathcal{U}, \widehat{H}^2_j(x)\rangle}.$$

We compute the numerator of the above expression:

$$\begin{aligned} < \mathcal{U}, x^2 \widehat{H}'_n(x)\widehat{H}_j(x) > &= -2 < \mathcal{U}, x\widehat{H}_n(x)\widehat{H}_j(x) > +2 < \mathcal{U}, x^3\widehat{H}_n(x)\widehat{H}_j(x) > \\ &\quad - < \mathcal{U}, x^2\widehat{H}_n(x)\widehat{H}'_j(x) > . \end{aligned}$$

For $j < n-3$ we get

$$< \mathcal{U}, x^2 \widehat{H}'_n(x)\widehat{H}_j(x) >= 0.$$

Thus if $j \leq n-4$ then

$$\lambda_{n,j} = 0.$$

Now we compute the coefficients $\lambda_{n,j}$ for $j \geq n-5$. Taking into account $\mathcal{U}$ is a symmetric linear functional

$$\lambda_{n,j} = 0, \qquad \text{for} \quad j = n, n-2.$$

Furthermore

$$\begin{aligned} \lambda_{n,n-3} &= 2\widehat{a}_n\widehat{a}_{n-1}\widehat{a}_{n-2}, \\ \lambda_{n,n-1} &= \widehat{a}_n\Big[-(n+1) + 2\big[\widehat{a}_{n+1} + \widehat{a}_n + \widehat{a}_{n-1}\big]\Big]. \end{aligned}$$

We compute the polynomials involved in the differential recurrence relation by using the TTRR in terms of $\widehat{H}_{n-1}(x)$ and $\widehat{H}_n(x)$. So the equation (12) can be reduced to

$$x^2\widehat{H}'_n(x) = \Big[2\widehat{a}_n x^2 + \widehat{a}_n\big[2(\widehat{a}_{n+1} + \widehat{a}_n) - (2n+1)\big]\Big]\widehat{H}_{n-1}(x) - \big[2\widehat{a}_n - n\big]x\widehat{H}_n(x).$$

If $A(x,n)$ denotes the coefficient of $\widehat{H}_{n-1}(x)$ and $B(x,n)$ denotes the coefficient of $\widehat{H}_n(x)$, i.e.

$$x^2\widehat{H}'_n(x) = A(x,n)\widehat{H}_{n-1}(x) - B(x,n)\widehat{H}_n(x),$$

and C_n is

$$C_n = 2\widehat{a}_n - n, \qquad n \geq 1,$$

then the differential recurrence relation becomes

$$x^2\widehat{H}'_n(x) = \widehat{a}_n\big[2x^2 + C_{n+1} + C_n\big]\widehat{H}_{n-1}(x) - C_n x\widehat{H}_n(x), \qquad n \geq 1,$$

and the Theorem holds. ■

Theorem 4 *For $n \geq 1$*

$$\begin{aligned} C_{2n} &= 1 - \frac{1}{1 + \dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}, && (13) \\ C_{2n+1} &= -1 + \frac{1}{1 + \left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}. && (14) \end{aligned}$$

Proof. Taking into account the definition of the coefficients C_n (11) we get

$$\begin{aligned} C_{2n} = 2\widehat{a}_{2n} - 2n &= 2n\left(1 + \frac{1}{2n} - \frac{\dfrac{1}{2n}}{1 + \dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}\right) - 2n \\ &= 1 - \frac{1}{1 + \dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}, \qquad n \geq 1, \end{aligned}$$

and

$$\begin{aligned} C_{2n+1} &= 2\widehat{a}_{2n+1} - (2n+1) = 2\left[\frac{2n+1}{2}\frac{1 + \dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1 + \left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}\right] - (2n+1) \\ &= (2n+1)\left[\frac{1 + \dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1 + \left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda} - 1\right] \\ &= \frac{2n+1}{2n}\frac{-\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1 + \left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}, \qquad n \geq 1, \end{aligned}$$

thus

$$\begin{aligned} C_{2n+1} &= \frac{-\left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}{1 + \left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda} \\ &= -1 + \frac{1}{1 + \left(1 + \dfrac{1}{2n}\right)\dfrac{(2n-1)!}{2^{2n-2}(n-1)!^2}\lambda}, \qquad n \geq 1. \end{aligned}$$

So the theorem holds. ■

Using the Stirling's formula, see [1],

$$x! = \sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x+\frac{\theta}{12x}}, \qquad 0 < x, \quad 0 < \theta < 1,$$

we get

$$(n-1)! \sim \frac{1}{n}n^n\sqrt{2n\pi}e^{-n}, \qquad n > 1,$$

so

$$
\begin{aligned}
\frac{(2n-1)!}{2^{2n-2}(n-1)!^2} &\sim \frac{1}{2^{2n-2}} \frac{\frac{1}{2n}(2n)^{2n}e^{-2n}\sqrt{4n\pi}}{\frac{1}{n^2}\left[n^n e^{-n}\sqrt{2n\pi}\right]^2} \\
&= \frac{1}{2^{2n-2}} \frac{2^{2n}\sqrt{n\pi}}{2\pi} = \frac{2\sqrt{\pi n}}{\pi}
\end{aligned}
$$

Then

$$\frac{(2n-1)!}{2^{2n-2}(n-1)!^2} \sim 2\sqrt{\frac{n}{\pi}}. \tag{15}$$

Thus from Theorem 4 and (15) we get

$$\lim_{n\to\infty} C_{2n} = 1, \tag{16}$$

and

$$\lim_{n\to\infty} C_{2n+1} = -1. \tag{17}$$

Let us denote $L_1(x,n)$ the lowering operator

$$L_1(x,n) = \left[x^2\frac{d}{dx} + B(x,n)\right]. \tag{18}$$

The statement of the previous theorem reads

$$L_1(x,n)\widehat{H}_n(x) = A(x,n)\widehat{H}_{n-1}(x). \tag{19}$$

Now we deduce some properties of the coefficients $A(x,n)$, $B(x,n)$, and C_n.

Lemma 5 *For $n \geq 1$, $A(x,n)$ and $B(x,n)$ satisfy*

$$\frac{1}{x}\Big[B(x,n+1) + B(x,n)\Big] = \frac{A(x,n)}{\widehat{a}_n} - 2x^2.$$

Theorem 6 *For $n \geq 1$*

(i) $\quad C_{2n} + C_{2n-1} = 0,$

(ii) $\quad \widehat{a}_{2n}\big(C_{2n+1} + C_{2n}\big) = \widehat{a}_{2n-1}\big(C_{2n-1} + C_{2n-2}\big).$

From the previous Theorem 6 we get

$$
\begin{aligned}
(C_{2n+1} + C_{2n}) &= \frac{\widehat{a}_{2n-1}}{\widehat{a}_{2n}}(C_{2n-1} - C_{2n-3}) \\
&= \frac{\widehat{a}_{2n-1}}{\widehat{a}_{2n}}\frac{\widehat{a}_{2n-3}}{\widehat{a}_{2n-2}}(C_{2n-3} - C_{2n-5}) \\
&\vdots \\
&= \frac{\widehat{a}_{2n-1}}{\widehat{a}_{2n}}\frac{\widehat{a}_{2n-3}}{\widehat{a}_{2n-2}}\cdots\frac{\widehat{a}_3}{\widehat{a}_4}(C_3 - C_1), \qquad n \geq 1.
\end{aligned}
$$

We show that $C_3 - C_1 < 0$. Indeed

$$C_3 - C_1 = 2\widehat{a}_3 - 3 - (2\widehat{a}_1 - 1) = 2(\widehat{a}_3 - \widehat{a}_1) - 3 + 1 = 2(\widehat{a}_3 - \widehat{a}_1 - 1).$$

On the other hand

$$\widehat{a}_3 - \widehat{a}_1 = \frac{3(1+\lambda)}{2+3\lambda} - \frac{1}{2(1+\lambda)} = \frac{6\lambda^2 + 9\lambda + 4}{6\lambda^2 + 10\lambda + 4}.$$

Obviously, since $0 \leq \lambda$ then $\widehat{a}_3 - \widehat{a}_1 \leq 1$ because we are dealing with a continuous function in λ. Thus we deduce that

$$C_{2n+1} + C_{2n} < 0, \qquad n \geq 0, \quad \lambda > 0, \tag{20}$$

and if $\lambda = 0$ then $C_{2n+1} + C_{2n} = 0$ for $n \geq 0$.

3.2 Raising operator

Using the TTRR and substituting it at the lowering operator then the raising operator for the orthogonal polynomials $\widehat{H}_n$ is deduced.

Theorem 7 *$\widehat{H}_n(x)$ satisfies*

$$\left[x^2 \frac{d}{dx} - \frac{A(x,n)}{\widehat{a}_n} x + B(x,n)\right] \widehat{H}_n(x) = \frac{-A(x,n)}{\widehat{a}_n} \widehat{H}_{n+1}(x).$$

Proof. From the TTRR we get

$$\widehat{H}_{n-1}(x) = \frac{x}{\widehat{a}_n} \widehat{H}_n(x) - \frac{1}{\widehat{a}_n} \widehat{H}_{n+1}(x).$$

Substituting in the expression of the lowering operator

$$x^2 \widehat{H}_n'(x) = A(x,n) \left[\frac{x}{\widehat{a}_n} \widehat{H}_n(x) - \frac{1}{\widehat{a}_n} \widehat{H}_{n+1}(x)\right] - B(x,n)\widehat{H}_n(x).$$

Thus our statement follows. ■

We will denote $L_2(x,n)$ the raising operator, i.e.

$$L_2(x,n) = \left[x^2 \frac{d}{dx} - \frac{A(x,n)}{\widehat{a}_n} x + B(x,n)\right].$$

4 Second-order linear differential equation

Combining raising and lowering operators we obtain a second order linear differential equation which is the key in order to give the electrostatic interpretation for the zero distribution of $\widehat{H}_n$. It is worth pointing out the role of the lowering operator since the coefficient $A(x,n)$ is involved in the second order linear differential equation.

Theorem 8 *$\widehat{H}_n(x)$ satisfies the following second order differential equation*

$$M(x,n)\widehat{H}_n''(x) + N(x,n)\widehat{H}_n'(x) + R(x,n)\widehat{H}_n(x) = 0, \qquad n \geq 0, \tag{21}$$

where

$$\begin{aligned} M(x,n) &= x^4 A(x,n), \\ N(x,n) &= -x^4 A'(x,n) + 2A(x,n)\Big[x^3 - x^5\Big], \end{aligned}$$

and $R(x,n)$ is explicitly given in terms of $A(x,n)$ and $B(x,n)$.

Proof. Taking into account the expression of the raising operator for $n-1$ as well as Lemma 5, we get

$$\left[x^2 \frac{d}{dx} - 4x^5 - B(x,n)\right] \widehat{H}_{n-1}(x) = \frac{-A(x,n-1)}{\widehat{a}_{n-1}} \widehat{H}_n(x). \tag{22}$$

Next from Theorem 3

$$\frac{1}{A(x,n)} \Big[x^2 \widehat{H}_n'(x) + B(x,n)\widehat{H}_n(x)\Big] = \widehat{H}_{n-1}(x),$$

we get

$$\begin{aligned} x^2\Big[-\frac{A'(x,n)}{A^2(x,n)}\big(x^2\widehat{H}_n'(x) + B(x,n)\widehat{H}_n(x)\big) \\ +\frac{1}{A(x,n)}\big(x^2\widehat{H}_n''(x) + (2x + B(x,n))\widehat{H}_n'(x) + B(x,n)\widehat{H}_n(x)\big)\Big] \\ -\frac{4x^5 + B(x,n)}{A(x,n)}\Big(x^2\widehat{H}_n'(x) + B(x,n)\widehat{H}_n(x)\Big) &= -\frac{A(x,n-1)}{\widehat{a}_{n-1}}\widehat{H}_n(x). \end{aligned}$$

Our statement follows with

$$R(x,n) = A(x,n)\left[-4x^5 + B'(x,n)x^2 + \frac{A(x,n-1)A(x,n)}{\widehat{a}_{n-1}}\right] - A'(x,n)B(x,n)x^2.$$

■

5 Electrostatic interpretation

In this section we propose an electrostatic model in the presence of a varying external potential from the second order linear differential equation deduced in the previous section. We will study the asymptotic behavior of the position of the fixed varying charges involved in the external potential. As we have shown in section 2 odd Hermite-type orthogonal polynomials coincide with the Hermite polynomials of odd degree. Thus the electrostatic interpretation for these polynomials comes from the electrostatics for Hermite polynomials. We denote $\{x_{n,k}\}_{1\leq k\leq n}$ the zeros of $\widehat{H}_n(x)$. Evaluating the second-order linear differential equation at $x_{n,k}$

$$M(x_{n,k},n)\widehat{H}_n''(x_{n,k})+N(x_{n,k},n)\widehat{H}_n'(x_{n,k})=0, \qquad 1\leq k\leq n.$$

Then

$$\begin{aligned} \frac{\widehat{H}_n''(x_{n,k})}{\widehat{H}_n'(x_{n,k})} &= -\frac{N(x_{n,k},n)}{M(x_{n,k},n)} \\ &= \frac{A'(x_{n,k},n)}{A(x_{n,k},n)}-\frac{2}{x_{n,k}}+2x_{n,k}, \qquad 1\leq k\leq n. \end{aligned} \tag{23}$$

We must point out that $A(x_{2n,k},2n)\neq 0$. Otherwise, from (3) we get $\widehat{H}_{2n}'(x_{2n,k})=0$ which is a contradiction because the zeros of the polynomials are simple. In the odd case, apparently it can be thought that we divide by zero because zero is a root of $\widehat{H}_{2n+1}(x)$ but the equation (23) can be reduced and then such a pathology does not appear. Indeed

$$\frac{A'(x_{2n+1,k},2n+1)}{A(x_{2n+1,k},2n+1)}-\frac{2}{x_{2n+1,k}}+2x_{2n+1,k}=2x_{2n+1,k}, \quad 1\leq k\leq n. \tag{24}$$

Applying the algebraic relation (see [7] and [11])

$$\frac{\widehat{H}_n''(x_{n,k})}{\widehat{H}_n'(x_{n,k})}=-2\sum_{j=1,j\neq k}^{n}\frac{1}{x_{n,j}-x_{n,k}},$$

the system (23) becomes

$$\sum_{j=1,j\neq k}^{n}\frac{1}{x_{n,j}-x_{n,k}}+\frac{A'(x_{n,k},n)}{2A(x_{n,k},n)}-\frac{1}{x_{n,k}}+x_{n,k}=0, \quad 1\leq k\leq n. \tag{25}$$

The total external potential $V(x)$ is the sum of an external field, $v(x)=x^2/2$, and a varying external potential, $\frac{1}{2}\ln\left|A(x,n)\right|-\ln|x|$, called by Ismail [7] long range field and short range field respectively, that is,

$$V(x)=\frac{x^2}{2}+\frac{1}{2}\ln\left|A(x,n)\right|-\ln|x|, \quad x\in\mathbb{R}-\{0\}. \tag{26}$$

We consider the potential energy at x of a point charge q located at t is $-q\ln|x-t|$. The zeros of the coefficient $A(x,n)$ play a very important role in the study of the electrostatic interpretation. We compute these zeros and we study the behavior of them since they will provide us the location of some fixed charges.

Corollary 9 *For $n \geq 1$, $A(x,n)$ has two real zeros.*

$$r_1(n) = \sqrt{-\frac{C_{n+1}+C_n}{2}}, \tag{27}$$

$$r_2(n) = -\sqrt{-\frac{C_{n+1}+C_n}{2}}. \tag{28}$$

Moreover, if n is even these zeros tend to the origin and if n is odd $r_1(n) = r_2(n) = 0$ is a double zero of $A(x,n)$.

Proof. It follows after the computation of the zeros of $A(x,n)$ and applying Theorem 4, (16) and (17). Obviously the zeros are real because of the property (20). ■

Let introduce the following electrostatic model:

> *Consider the system of n movable unit positive charges in n distinct points $\{x_{n,i}\}_{i=1}^{n}$ of the real line in the presence of the total external potential $V(x)$.*

Notice that from (26) we can deduce that the zeros of $A(x,n)$ give us the position depending on n of two fixed charges. Then the external field is generated by two fixed charges $-1/2$ at the real positions $r_1(n)$, $r_2(n)$, as well as a charge $+1$ at zero is fixed because of the perturbation of the weight function. When n tends to infinity $r_1(n)$ and $r_2(n)$ tend to the origin, then they will be cancelled with the charge $+1$ at the origin. So it will be the classical electrostatic interpretation at the limit. In the odd case, we know $A(x,2n-1)$ is reduced because of Theorem 6, then it follows the external field is generated by one fixed charge at the origin. It is worthy of pointing out the charge -1 from the varying external potential at the origin and the charge $+1$ because of the mass point at the origin are cancelled each other. Again this case coincides with the classical electrostatic interpretation.

We denote the position vector as

$$\mathbf{x} = (x_{n,1}, x_{n,2}, ..., x_{n,n}),$$

where $x_{n,j} < x_{n,k}$ if $j < k$. The total energy of the system is

$$E(\mathbf{x}) = \sum_{k=1}^{n} V(x_{n,k}) - \sum_{1\leq j<k\leq n} \ln|x_{n,j} - x_{n,k}|.$$

Notice that (25) is the derivative of the energy function. This means the zeros of the Hermite-type orthogonal polynomials are critical points of the energy function. Now studying the Hessian matrix

$$H = (h_{i,j}), \quad h_{i,j} = \frac{\partial^2 E}{\partial x_i \partial x_j},$$

we will deduce when $E(x)$ gets local minima at the zeros of the polynomials $\widehat{H}_n(x)$. Indeed, taking into account

$$h_{k,l} = \begin{cases} \dfrac{1}{2}\dfrac{\partial}{\partial x_{n,k}}\left(\dfrac{A'(x_{n,k},n)}{A(x_{n,k},n)}\right) + \dfrac{1}{x_{n,k}^2} + 1 + \displaystyle\sum_{j=1,j\neq k}^{n} \dfrac{1}{(x_{n,j}-x_{n,k})^2}, & \text{if } k=l, \\ \\ -\dfrac{1}{(x_{n,l}-x_{n,k})^2}, & \text{if } k\neq l, \end{cases} \tag{29}$$

the Hessian matrix is real and symmetric. If we obtain conditions in order to the Hessian matrix be strictly diagonally dominant and its diagonal terms are positive then H will be positive definite, see [5, Corollary (7.2.3)]. In this case, we will have conditions in order to the equilibrium position of the proposed system will be reached at the zeros of $\widehat{H}_n(x)$. Thus, we need to guarantee the following function is positive

$$\frac{1}{2}\frac{\partial}{\partial x_{n,k}}\left(\frac{A'(x_{n,k},n)}{A(x_{n,k},n)}\right) + \frac{1}{x_{n,k}^2} + 1, \qquad 1\leq k\leq n. \tag{30}$$

So we study the function

$$f(x,n) = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{A'(x,n)}{A(x,n)}\right) + \frac{1}{x^2} + 1 \tag{31}$$

In the odd case, from Theorem 6

$$\begin{aligned} f(x,2n+1) &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{A'(x,2n+1)}{A(x,2n+1)}\right) + \frac{1}{x^2} + 1 \\ &= 4\frac{-2x^2+C_{2n+2}+C_{2n+1}}{(2x^2+C_{2n+2}+C_{2n+1})^2} + \frac{1}{x^2} + 1 = 1. \end{aligned} \tag{32}$$

In the even case, we study the asymptotic behavior of $f(x,2n)$

$$\begin{aligned} f(x,2n) &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{A'(x,2n)}{A(x,2n)}\right) + \frac{1}{x^2} + 1 \\ &= 4\frac{-2x^2+C_{2n+1}+C_{2n}}{(2x^2+C_{2n+1}+C_{2n})^2} + \frac{1}{x^2} + 1 \\ &= \frac{4x^6+4(C_{2n+1}+C_{2n})x^4+\left(6(C_{2n+1}+C_{2n})+(C_{2n+1}+C_{2n})^2\right)x^2+(C_{2n+1}+C_{2n})^2}{(2x^2+C_{2n+1}+C_{2n})^2x^2} \end{aligned} \tag{33}$$

For n sufficiently large and using (16) and (17) we conclude

$$f(x,2n)\sim 1.$$

For n large enough the Hessian matrix is positive definite and then the electrostatic equilibrium position in the presence of the external field, $V(x)$ is obtained at the zeros $\{x_{n,i}\}_{i=1}^{n}$

of the Hermite-type orthogonal polynomial $\widehat{H}_n$, provided that the particle interaction obeys a logarithmic potential. Otherwise we cannot assert that one obtains electrostatic equilibrium at the zeros of the orthogonal polynomials.

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